Extended Rayleigh-Schrödinger perturbation theory for the quartic anharmonic oscillator

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Abstract
This paper follows the ideas of the author's earlier works that there is a possible metastable process passing from unperturbed to perturbed as a perturbation potential evolves. Furthermore, we extend the balance condition which was fully described in earlier works, from the total wave to each corresponding sth-order component wave $\Psi_n^{(s)}$, $s = 0, 1, \ldots$. Therefore, to our great benefit, an infinite set of coupled balance conditions can be gained. We add this new idea to perturbation theory, and hence present the so-called extended Rayleigh–Schrödinger perturbation theory. As is well known, earlier applications in PFSKB (principles of the first and second kind of balance) theory seldom went beyond the first to, at most, second order in the eigenvalue because of computational difficulties in evaluating the minimization of the total energy of infinite summations which appear in the equations for the higher-order terms. Fortunately, these difficulties can be overcome by individually balancing the conditions of the corresponding sth-order component waves $\Psi_n^{(s)}$. However, it seems reasonable to hope that a better understanding of extended Rayleigh–Schrödinger perturbation theory may be gained. In many cases of interest, the quartic anharmonic oscillator is chosen as an example for the demonstration of extended Rayleigh–Schrödinger perturbation theory because of its divergent Rayleigh–Schrödinger perturbation expansions and wide applications.

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1. Theory

We investigate a quantum system (in particular, an atomic and molecular system), which acts under a perturbing potential H'. We are interested in a certain effect in the unperturbed system which is caused by the perturbing potential H': an isolated unperturbed wave $\Psi_n^{(0)}$ and perturbing wave $\phi_{\cdot,\cdot}^{(p)}$ move closely, the perturbing wave $\phi_n^{(p)}$ imposes a force on the unperturbed wave $\Psi_n^{(0)}$, which may then vary its structure parameters α to balance this external force and, after a very short period of time, the total wave $\Psi_n(x, \alpha_n^*)$ stabilizes at a balance position

with minimal energy. In other words, a certain possible metastable process takes place if the perturbing potential field is present.

As is well known, we assume in PFSKB (principles of the first and second kind of balance) theory that the total metastable wave in the possible metastable process can be described by an assumed extended Schrödinger equation (the details of the method are given in [1–3]), namely

$$H\Psi_n(x, \alpha(t)) = E_n(\alpha)\Psi_n(x, \alpha(t)) \qquad t_i \leqslant t \leqslant t_f \tag{1}$$

where α is a structure parameter of the zeroth-order wavefunction, that varies implicitly with time t.

Ideally, one would solve equation (1) analytically so as to obtain the metastable wavefunction $\Psi_n(x, \alpha)$, which is necessarily an explicit function of the parameter α. Unfortunately, in all but the simplest cases, however, it seems usually to be impossible or impractical to do so. However, fortunately, the extended Schrödinger equation (1) can be solved by extended Rayleigh–Schrödinger (ERS) perturbation theory. This paper presents an intensive study of ERS perturbation theory, and attempts to establish all the relevant formulae in detail. Finally, we work through the procedure of this so-called ERS perturbation theory for one of the most interesting cases: the quartic anharmonic oscillator.

In this section, we summarize the necessary background for the present study. The point of departure for ERS perturbation theory is the well known extended perturbation Hamiltonian $H_1(\alpha)$. We define the difference of the total Hamiltonian H and the varying zeroth-order Hamiltonian $H_0(\alpha)$ to be chosen as the extended perturbation Hamiltonian:

$$H_1(\alpha) = H - H_0(\alpha) = H_0(C) - H_0(\alpha) + H' \tag{2}$$

where $H_0(C)$ is an initial unperturbed Hamiltonian and H' is a conventional perturbation.

It is sufficient to establish all the essential details of the formulae of this so-called ERS perturbation theory in a similar way to the standard RS perturbation theory. The main steps are shown below. Considering the similarity with RS theory, it is straightforward to obtain the mth-order approximation to an energy level

$$E_n^{\langle m \rangle}(\alpha) = E_n^{(0)}(\alpha) + \sum_{s=1}^{m} E_n^{(s)}(\alpha) \tag{3}$$

with

$$E_n^{(s)}(\alpha) = \langle \Psi_n^{(0)}(\alpha)|H_1(\alpha)|\Psi_n^{(s-1)}(x, \alpha)\rangle \qquad s = 1, 2, \ldots. \tag{4}$$

Correspondingly, the sth-order perturbing wavefunction, i.e. the sth-order component wave

$$\Psi_n^{(s)}(x, \alpha) = \sum_{j \neq n} C_{nj}^{(s)} \Psi_j^{(0)}(x, \alpha) \tag{5}$$

where the expansion coefficients are denoted by

$$C_{nj}^{(s)} = \frac{1}{E_n^{(0)}(\alpha) - E_j^{(0)}(\alpha)} \left\{ \sum_{k \neq n} C_{nk}^{(s-1)}(H_1)_{jk} - \sum_{i=1}^{s-1} E_n^{(i)} C_{nj}^{(s-i)} \right\}. \tag{6}$$

We find to our surprise that one can recover the standard RS perturbation just by setting $\alpha = C$ in equations (3)–(6).

In this work a new idea of balancing is developed: we assume that balance is realized for each sth-order component wave, $\Psi_n^{(s)}$, $s = 0, 1, \ldots$, and thus an infinite set of coupled differential equations can be found:

$$\langle \Psi_n^{(0)}(x, \alpha)|H_1(\alpha)|\Psi_n^{(s)}(x, \alpha)\rangle \big|_{\alpha=\alpha_n^*} = 0 \qquad s = 0, 1, 2 \ldots \tag{7}$$

for the first kind of balance, and

$$\frac{d}{d\alpha}\langle\Psi_n^{(0)}(x,\alpha)|H|\Psi_n^{(s)}(x,\alpha)\rangle\big|_{\alpha=\alpha_n^*} = 0 \qquad s = 0, 1, 2\ldots \tag{8}$$

for the second kind of balance. We note that the balance conditions, equations (7) and (8), correspond to each sth-order component wave, $\Psi_n^{(s)}$, $s = 0, 1, \ldots$ of the total wave $\Psi_n(x,\alpha)$.

Consideration of the orthonormal condition allows us to rewrite equation (8) as follows:

$$\frac{d}{d\alpha}E_n^{\langle 1\rangle}(\alpha)\bigg|_{\alpha=\alpha_n^{(1)}} = 0 \qquad \text{for} \quad s = 0 \tag{9}$$

and

$$\frac{d}{d\alpha}E_n^{(s+1)}(\alpha)\bigg|_{\alpha=\alpha_n^{(s+1)}} = 0 \qquad \text{for} \quad s = 1, 2, \ldots \tag{10}$$

where $E_n^{\langle 1\rangle}$ is the first-order approximation to energy and $E_n^{(s+1)}$ is the $(s+1)$th-order perturbation energy. We call attention to the new idea developed in this work that if the ith-order component wave is in the first kind of balance, the contribution to the energy level of the ith-order component wave will be zero, namely

$$E_n^{(i+1)}\big|_{\alpha=\alpha_n^{(i+1)}} = 0 \tag{11}$$

and so

$$E_n^{\langle i+1\rangle}\big|_{\alpha=\alpha_n^{(i+1)}} = E_n^{\langle i\rangle} \tag{12}$$

where $E_n^{\langle i+1\rangle}$ and $E_n^{\langle i\rangle}$ correspond to the $(i + 1)$th- and ith-order approximation to energy.

Obviously, the zeroth-order wave $\Psi_n^{(0)}(x,\alpha)$ is the principal part of the total wave $\Psi_n(x,\alpha)$, and so the first-order approximation to energy $E_n^{\langle 1\rangle}$ is the dominant term of the exact energy, namely

$$E_n^{\langle 1\rangle} \gg E_n^{(p)} \tag{13}$$

and

$$\Psi_n^{(0)} \gg \phi_n^{(p)} \tag{14}$$

where $E_n^{(p)}$ represents contributions from the total perturbing wave $\phi_n^{(p)}$. We believe that the zeroth-order wave $\Psi_n^{(0)}(x,\alpha)$ characterizes the behaviour of the exact wave $\Psi_n(x,\alpha)$. According to PESKB theory, as the metastable process finishes, the total wave is to be balanced in stationary states with minimal energy. It appears reasonable that one more condition should be added to the zeroth-order wave:

$$\frac{d^2}{d\alpha^2}E_n^{\langle i\rangle}(\alpha)\big|_{\alpha=\alpha_n^{(1)}} > 0. \tag{15}$$

It is important that the additional condition (15) ensures stability of the total wave. This is a necessary requirement for physical reality.

We restrict ourselves here to the important case of an anharmonic oscillator, as this is sufficient to demonstrate all the essential details of this so-called ERS perturbation theory. We shall do this in the next section.

2. Calculation

A study of the quartic anharmonic oscillator is undoubtably of considerable interest for evaluating ERS perturbation theory; since the early works of Bender and Wu [4–8] it has been realized that the standard RS perturbation series diverged in this case for all values of the coupling parameter, no matter how small the coupling was.

As is well known, the quantum anharmonic oscillator is described by the time-independent Schrödinger equation [13–18]

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 + \delta x^4\right)\Psi_n(x) = E_n\Psi_n(x). \tag{16}$$

It follows from equation (16) that the extended Schrödinger equation takes the form

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{\omega^2 x^2}{2} + \frac{1-\omega^2}{2}x^2 + \delta x^4\right)\Psi_n(x,\omega) = E_n(\omega)\Psi_n(x,\omega) \tag{17}$$

with the zeroth-order equation

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\omega^2 x^2\right)\Psi_n^{(0)}(x,\omega) = (n+\tfrac{1}{2})\omega\Psi_n^{(0)}(x,\omega) \tag{18}$$

and

$$H_1(\omega) = \frac{1-\omega^2}{2}x^2 + \delta x^4 \tag{19}$$

where $H_1(\omega)$ is the extended perturbation, $\frac{1-\omega^2}{2}x^2$ is called the 'deformation' term of the zeroth-order Hamiltonian $H_0(\omega)$ with varying structure parameter ω, and ω is an oscillation frequency.

2.1. First-order energy correction

We now wish to obtain the first-order energy correction from equation (4), it follows that

$$E_n^{(1)}(\omega) = \left(n+\tfrac{1}{2}\right)\frac{1-\omega^2}{2\omega} + \frac{3\delta}{2\omega^2}\left(n^2+n+\tfrac{1}{2}\right). \tag{20}$$

Moreover, the first-order energy approximation

$$E_n^{\langle 1 \rangle}(\omega) = \left(n+\tfrac{1}{2}\right)\frac{1+\omega^2}{2\omega} + \frac{3\delta}{2\omega^2}\left(n^2+n+\tfrac{1}{2}\right). \tag{21}$$

According to the balance condition (9) of the zeroth-order wave $\Psi_n^{(0)}$ and an additional minimization requirement (15), a cubic equation follows:

$$\omega^3 + p_n\omega + q_n = 0 \tag{22}$$

with

$$p_n = -1 \qquad q_n = -\frac{6\delta\left(n^2+n+\tfrac{1}{2}\right)}{n+\tfrac{1}{2}}. \tag{23}$$

Suppose that $\Omega_n^{(1)}$ is the unique real root of equation (22), we have the first-order approximate balanced energy level

$$E_n^{\langle 1 \rangle} = (n+\tfrac{1}{2})\frac{3\left(\Omega_n^{(1)}\right)^2+1}{4\Omega_n^{(1)}} \qquad n = 0, 1, \ldots. \tag{24}$$

It is easy to find the real root for the standard cubic equation (22) using the algebraic formula

$$\Omega_n^{(1)} = \left\{ -\frac{q_n}{2} + \left[\left(\frac{q_n}{2} \right)^2 + \left(\frac{p_n}{3} \right)^3 \right]^{1/2} \right\}^{1/3} + \left\{ -\frac{q_n}{2} - \left[\left(\frac{q_n}{2} \right)^2 + \left(\frac{p_n}{3} \right)^3 \right]^{1/2} \right\}^{1/3}. \tag{25}$$

To estimate the accuracy of our first-order approximation of the ERS perturbation, we compare our results with the exact numerical calculation carried out by Schwartz and Simon [19] for the ground-state energy E_0—our result agrees with the exact numerical result to within 2% for the full range of δ between 0 and ∞ [20]. From equation (22), the asymptotic value of δ for $\delta \to \infty$ is

$$\Omega_n^{(1)} = \left[\frac{6\delta \left(n^2 + n + \frac{1}{2} \right)}{n + \frac{1}{2}} \right]^{1/3} \qquad n = 0, 1, \ldots \tag{26}$$

with

$$E_n^{\langle 1 \rangle}(\delta \to \infty) = \tfrac{3}{4} \left(n + \tfrac{1}{2} \right) \Omega_n^{(1)}(\delta \to \infty) \qquad n = 0, 1, \ldots . \tag{27}$$

In particular, for the ground state, our result predicts

$$E_0^{\langle 1 \rangle}(\delta \to \infty) = \tfrac{3}{8}(6\delta)^{1/3} = 0.681\,42\,\delta^{1/3} \tag{28}$$

while the exact answer is

$$E_0^{\text{exact}}(\delta \to \infty) = 0.667\,99\,\delta^{1/3}. \tag{29}$$

The cubic equation (22) has a unique real root, so that the minimum of $E_n^{\langle 1 \rangle}$ is unique as well.

2.2. The second-order perturbation energy

$$E_n^{(2)}(\omega) = (H_1)_{n,n+4} C_{n,n+4}^{(1)} + (H_1)_{n,n+2} C_{n,n+2}^{(1)} + (H_1)_{n,n-2} C_{n,n-2}^{(1)} + (H_1)_{n,n-4} C_{n,n-4}^{(1)} \tag{30}$$

with

$$(H_1)_{n,n+4} = \frac{\delta}{4\omega^2} \sqrt{(n+1)(n+2)(n+3)(n+4)} \tag{31a}$$

$$(H_1)_{n,n+2} = \frac{1}{\omega} \sqrt{(n+1)(n+2)} \left[\frac{1 - \omega^2}{4} + \frac{\delta}{\omega} \left(n + \tfrac{3}{2} \right) \right] \tag{31b}$$

$$(H_1)_{n,n-2} = \frac{1}{\omega} \sqrt{n(n-1)} \left[\frac{1 - \omega^2}{4} + \frac{\delta}{\omega} \left(n - \tfrac{1}{2} \right) \right] \tag{31c}$$

$$(H_1)_{n,n-4} = \frac{\delta}{4\omega^2} \sqrt{n(n-1)(n-2)(n-3)} \tag{31d}$$

$$C_{n,n+4}^{(1)} = -\frac{1}{4\omega} (H_1)_{n,n+4} \tag{31e}$$

$$C_{n,n+2}^{(1)} = -\frac{1}{2\omega} (H_1)_{n,n+2} \tag{31f}$$

$$C_{n,n-2}^{(1)} = \frac{1}{2\omega} (H_1)_{n,n-2} \tag{31g}$$

$$C_{n,n-4}^{(1)} = \frac{1}{4\omega} (H_1)_{n,n-4}. \tag{31h}$$

On substitution of equations (31) into equation (30), we obtain an explicit expression for the second-order perturbation energy:

$$E_n^{(2)}(\omega) = \frac{\delta^2}{\omega^5} A + \frac{1 - \omega^2}{\omega^4} B + \frac{\left(1 - \omega^2\right)^2}{\omega^3} C \tag{32a}$$

with

$$A = \tfrac{1}{64}[n(n - 1)(n - 2)(n - 3) - (n + 1)(n + 2)(n + 3)(n + 4)]$$
$$+ \tfrac{1}{2}\left[n(n - 1)\left(n - \tfrac{1}{2}\right)^2 - (n + 1)(n + 2)\left(n + \tfrac{3}{2}\right)^2\right] \tag{32b}$$

$$B = \tfrac{1}{4}\left[n(n - 1)\left(n - \tfrac{1}{2}\right) - (n + 1)(n + 2)\left(n + \tfrac{3}{2}\right)\right] \tag{32c}$$

$$C = \tfrac{1}{32}[n(n - 1) - (n + 1)(n + 2)]. \tag{32d}$$

The balance condition of the first-order component wave $\Psi_n^{(1)}$ yields

$$\left.\frac{\mathrm{d}E_n^{(2)}}{\mathrm{d}\omega}\right|_{\omega=\Omega_n^{(2)}} = 0. \tag{33}$$

The unique positive root $\Omega_n^{(2)}$ of equation (33) can be found easily by a numerical procedure [21], so that $E_n^{(2)}$ has a unique maximum because of

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}\omega^2} E_n^{(2)}\right|_{\omega=\Omega_n^{(2)}} < 0.$$

2.3. The third-order perturbation energy

$$E_n^{(3)}(\omega) = (H_1)_{n,n+4}C_{n,n+4}^{(2)} + (H_1)_{n,n+2}C_{n,n+2}^{(2)} + (H_1)_{n,n-2}C_{n,n-2}^{(2)} + (H_1)_{n,n-4}C_{n,n-4}^{(2)} \tag{34}$$

and

$$C_{n,n+4}^{(2)} = -\frac{1 - \omega^2}{4\omega^2}\left[\tfrac{1}{4}\sqrt{(n + 3)(n + 4)}\,C_{n,n+2}^{(1)} + 8C_{n,n+4}^{(1)}\right]$$
$$- \frac{\delta}{\omega^3}\left[\tfrac{1}{4}\sqrt{(n + 3)(n + 4)}\left(n + 2 + \tfrac{3}{2}\right)C_{n,n+2}^{(1)} + \frac{3\delta}{2}(2n + 5)C_{n,n+4}^{(1)}\right] \tag{35a}$$

$$C_{n,n+2}^{(2)} = -\frac{1 - \omega^2}{4\omega}\left[\tfrac{1}{2}\sqrt{(n + 3)(n + 4)}\,C_{n,n+4}^{(1)} + 2C_{n,n+2}^{(1)}\right]$$
$$+ \frac{\delta}{\omega^2}\left[-\tfrac{1}{2}\left(n + 4 - \tfrac{1}{2}\right)\sqrt{(n + 4)(n + 3)}\,C_{n,n+4}^{(1)}\right.$$
$$\left. - \tfrac{1}{8}\sqrt{(n - 1)n(n + 1)(n + 2)}\,C_{n,n-2}^{(1)} - \frac{3\delta}{2}(2n + 3)C_{n,n+2}^{(1)}\right] \tag{35b}$$

$$C_{n,n-2}^{(2)} = \frac{1 - \omega^2}{4\omega}\left[\tfrac{1}{2}\sqrt{(n - 3)(n - 2)}\,C_{n,n-4}^{(1)} - 2C_{n,n-2}^{(1)}\right]$$
$$+ \frac{\delta}{\omega^3}\left[\tfrac{1}{2}\sqrt{(n - 3)(n - 2)}\left(n - 4 + \tfrac{3}{2}\right)C_{n,n-4}^{(1)}\right.$$
$$\left. + \tfrac{1}{8}\sqrt{(n + 2)(n + 1)n(n - 1)}\,C_{n,n+2}^{(1)} - \frac{3\delta}{2}(2n - 1)C_{n,n-2}^{(1)}\right] \tag{35c}$$

$$C_{n,n-4}^{(2)} = \frac{1 - \omega^2}{4\omega^2}\left[\tfrac{1}{4}\sqrt{(n - 2)(n - 3)}\,C_{n,n-2}^{(1)} - 2C_{n,n-4}^{(1)}\right]$$
$$+ \frac{\delta}{\omega^3}\left[\tfrac{1}{4}\sqrt{(n - 2)(n - 3)}\left(n - 2 - \tfrac{1}{2}\right)C_{n,n-2}^{(1)} - \frac{3\delta}{2}(2n - 3)C_{n,n-4}^{(1)}\right]. \tag{35d}$$

We have scanned the third-order perturbing energy $E_n^{(3)}(\omega)$ and found that $E_n^{(3)}(\omega)$ varies monotonically, and has a unique zero point for the ground and first excited states, and both a minimum and maximum for higher excited states as well. This behaviour of $E_n^{(3)}(\omega)$ shows that the second-order component wave $\Psi_n^{(2)}$ is in the first kind of balance for $n = 0, 1$ states, and in the second kind of balance with two balancing positions for higher excited states of $n \geqslant 2$.

The balance conditions of the second-order component wave $\Psi_n^{(2)}$ yield

$$E_n^{(3)}(\omega)\big|_{\omega=\Omega_n^{(3)}(0)} = 0 \qquad \text{for} \quad n = 0, 1 \tag{36}$$

where roots of $\Omega_n^{(3)}(0)$ correspond to balance points of the first kind of balance, and

$$\frac{\mathrm{d}}{\mathrm{d}\omega} E_n^{(3)}(\omega)\big|_{\omega=\Omega_n^{(3)}(+),\Omega_n^{(3)}(-)} = 0 \qquad \text{for} \quad n \geqslant 2 \tag{37}$$

where roots of $\Omega_n^{(3)}(+)$ and $\Omega_n^{(3)}(-)$ correspond to the maximum and minimum, respectively. Routinely, the positive root $\Omega_n^{(3)}(+)$, $\Omega_n^{(3)}(-)$ and $\Omega_n^{(3)}(0)$ of equation (36) and (37) can be found by a numerical procedure.

2.4. The fourth-order perturbation energy

$$E_n^{(4)}(\omega) = (H_1)_{n,n+4}C_{n,n+4}^{(3)} + (H_1)_{n,n+2}C_{n,n+2}^{(3)} + (H_1)_{n,n-2}C_{n,n-2}^{(3)} + (H_1)_{n,n-4}C_{n,n-4}^{(3)} \tag{38}$$

with

$$C_{n,n+4}^{(3)} = -\frac{1}{4\omega}\big\{C_{n,n+2}^{(2)}(H_1)_{n+2,n+4} + C_{n,n+6}^{(2)}(H_1)_{n+6,n+4} + C_{n,n+8}^{(2)}(H_1)_{n+8,n+4}$$

$$+ C_{n,n+4}^{(2)}\big[(H_1)_{n+4,n+4} - E_n^{(1)}\big] - E_n^{(2)}C_{n,n+4}^{(1)}\big\} \tag{39a}$$

$$C_{n,n+2}^{(3)} = -\frac{1}{2\omega}\big\{C_{n,n+4}^{(2)}(H_1)_{n+4,n+2} + C_{n,n+6}^{(2)}(H_1)_{n+6,n+2} + C_{n,n-2}^{(2)}(H_1)_{n-2,n+2}$$

$$+ C_{n,n+2}^{(2)}\big[(H_1)_{n+2,n+2} - E_n^{(1)}\big] - E_n^{(2)}C_{n,n+2}^{(1)}\big\} \tag{39b}$$

$$C_{n,n-2}^{(3)} = \frac{1}{2\omega}\big\{C_{n,n-4}^{(2)}(H_1)_{n-4,n-2} + C_{n,n-6}^{(2)}(H_1)_{n-6,n-2} + C_{n,n+2}^{(2)}(H_1)_{n+2,n-2}$$

$$+ C_{n,n-2}^{(2)}\big[(H_1)_{n-2,n-2} - E_n^{(1)}\big] - E_n^{(2)}C_{n,n-2}^{(1)}\big\} \tag{39c}$$

$$C_{n,n-4}^{(3)} = \frac{1}{4\omega}\big\{C_{n,n-2}^{(2)}(H_1)_{n-2,n-4} + C_{n,n-6}^{(2)}(H_1)_{n-6,n-4} + C_{n,n-8}^{(2)}(H_1)_{n-8,n-4}$$

$$+ C_{n,n-4}^{(2)}\big[(H_1)_{n-4,n-4} - E_n^{(1)}\big] - E_n^{(2)}C_{n,n-4}^{(1)}\big\} \tag{39d}$$

and the second kind of balance condition yields

$$\frac{\mathrm{d}}{\mathrm{d}\omega} E_n^{(4)}(\omega) = 0. \tag{40}$$

The fourth-order perturbing energy $E_n^{(4)}(\omega)$ has been scanned with respect to ω for δ from 10^{-3} to 10^4, and n from 0 to 40. We find a unique maximum for the first four states and one maximum and one minimum if a particle is in higher excited states of $n \geqslant 5$.

3. Discussion and conclusions

The principal advantages of the ERS perturbation, unlike the RS perturbation, are the following.

(a) We assume a certain possible metastable process exists, and so yields an infinite set of powerful balance conditions which correspond to each component wave, $\Psi_n^{(0)}$, $\Psi_n^{(1)}$ and $\Psi_n^{(s)}, \ldots$.

(b) The total balance wave has a set of balance points $\{\Omega_n^{(s)}\}$, namely

$$\Psi_n(x, \Omega_n^{(1)}, \Omega_n^{(2)}, \ldots) = \Psi_n^{(0)}(x, \Omega_n^{(1)}) + \Psi_n^{(1)}(x, \Omega_n^{(2)}) + \cdots + \Psi_n^{(s)}(x, \Omega_n^{(s+1)}) + \cdots \quad (41)$$

and

$$\Psi_n^{(s)}(x, \Omega_n^{(s+1)}) = \sum_{j \neq n} C_{nj}^{(s)} \Psi_n^{(0)}(x, \Omega_n^{(s+1)}) \quad (42)$$

where an infinite number of balancing points $\{\Omega_n^{(s)}\}$ can be determined by corresponding balance equations.

(c) The ERS perturbation creates an infinite set of complete orthonormal wavefunction bases $\{\Psi_n^{(0)}(x, \Omega_n^{(s)})\}$. In general, there is one-to-one correspondence between the perturbing wave $\Psi_n^{(s-1)}$ and function basis $\{\Psi_n^{(0)}(x, \Omega_n^{(s)})\}$. Unlike in RS perturbation theory we only need one wavefunction basis which corresponds to all $\Omega_n^{(s)}$, taking an initial value of the structure parameter $\alpha = c$.

(d) It comes as no surprise that one can recover RS perturbation by just setting $\omega = c$ and removing the balance equations.

(e) Consideration of the limit of $H' \to 0$, but not equal to zero, straightforwardly yields that all balance points

$$\lim_{H' \to 0} \Omega_n^{(s)} = 1 \qquad s = 1, 2, \ldots. \quad (43)$$

So in this limit, the results of RS are close to the results of ERS. This is why one may consider the standard RS perturbation theory as inferior to ERS perturbation theory. It emerges in table 2 for a coupling parameter $\delta = 0.0001$ in the case of an anharmonic oscillator.

(f) As is well known, in many cases of interest, the perturbation H' may be so large that the ratio $E_n^{(1)}/E_n^{(0)} \gg 1$. As is to be anticipated, the results of the RS perturbation fail completely, but the results of the ERS perturbation appear to be reasonable [1–3]. This character is also shown in our example. For the case of an anharmonic oscillator the ratio

$$\frac{E_n^{(1)}}{E_n^{(0)}} = \frac{3\delta}{2} \frac{n^2 + n + \frac{1}{2}}{n + \frac{1}{2}}. \quad (44)$$

It is proportional to the coupling parameter δ and the state index n. As $\delta > 1$ and a particle in a highly excited state n, the ratio of $E_n^{(1)}/E_n^{(0)}$ will get far greater than one (see table 3 for the cases of coupling parameter $\delta = 1000$ and $10\,000$). One may anticipate that the results from the RS calculations will fail completely, while the ERS calculations appear to be reasonable.

In this work, we have carried out calculations of ERS perturbation up to a fourth-order approximation for highly excited states up to $n = 40$ and for a wide range of the coupling parameter, $\delta = 0.001$–$10\,000$. The comparison is made in table 1 up to the third-order approximation of ERS calculations with exact eigenvalues [9]. Here in tables 1–6 we use the notation $(+)$ and $(-)$ corresponding to the maximum and minimum for the second kind of

Table 1. Comparison of up to third-order ERS computed eigenvalues with exact values for states ($n = 0$–6) with coupling parameter $\delta = 0.1$–1000 and accuracy is estimated to ± 1 in the last digit in $m = \hbar = 1$ units. The notation of (+), (−) and (0) represents the balance state for the corresponding component waves.

δ	n	E_n^{exact}	$E_n^{(3)}(-)$	$E_n^{(3)}(+)$	$E_n^{(3)}(-)$	$E_n^{(3)}(0)$	$E_n^{(3)}(+)$	$E_n^{(2)}(+)$	$E_n^{(2)}(+)$	$E_n^{(1)}(-)$
0.1	0	0.559146	0.55895E+00	0.55895E+00		0.00000E+00		0.55895E+00	-0.13528E-02	0.56031E+00
	1	1.769503	0.17689E+01	0.17689E+01		0.00000E+00		0.17689E+01	-0.45084E-02	0.17734E+01
	2	3.138624	0.31387E+01	0.31394E+01	0.17469E-02		0.24785E-02	0.31369E+01	-0.13163E-02	0.31382E+01
	3	4.628883	0.46292E+01	0.46332E+01	0.26657E-02		0.66902E-02	0.46265E+01	0.45935E-02	0.46219E+01
	4	6.220300	0.62205E+01	0.62296E+01	0.30716E-02		0.12195E-01	0.62174E+01	0.12249E-01	0.62052E+01
	5		0.78996E+01	0.79152E+01	0.31668E-02		0.18754E-01	0.78965E+01	0.21237E-01	0.78752E+01
	6		0.96572E+01	0.96803E+01	0.30659E-02		0.26218E-01	0.96541E+01	0.31330E-01	0.96228E+01
0.5	0	0.696176	0.69489E+00	0.69489E+00		0.00000E+00		0.69489E+00	-0.67717E-02	0.70166E+00
	1	2.324406	0.23212E+01	0.23212E+01		0.00000E+00		0.23212E+01	-0.17942E-01	0.23391E+01
	2	4.327525	0.43275E+01	0.43293E+01	0.84487E-02		0.10227E-01	0.43191E+01	-0.44562E-02	0.43235E+01
	3	6.578400	0.65802E+01	0.65922E+01	0.11466E-01		0.23434E-01	0.65687E+01	0.13999E-01	0.65547E+01
	4	9.028779	0.90305E+01	0.90572E+01	0.11956E-01		0.38621E-01	0.90185E+01	0.34702E-01	0.89838E+01
	5		0.11649E+02	0.11693E+02	0.11385E-01		0.55341E-01	0.11638E+02	0.57016E-01	0.11581E+02
	6		0.14417E+02	0.14480E+02	0.10341E-01		0.73371E-01	0.14406E+02	0.80699E-01	0.14326E+02
1.0	0	0.803771	0.80161E+00	0.80161E+00		0.00000E+00		0.80161E+00	-0.10892E-01	0.81250E+00
	1	2.737892	0.27328E+01	0.27328E+01		0.00000E+00		0.27328E+01	-0.27175E-01	0.27599E+01
	2	5.179292	0.51791E+01	0.51814E+01	0.13237E-01		0.15518E-01	0.51659E+01	-0.64928E-02	0.51724E+01
	3	7.942400	0.79453E+01	0.79621E+01	0.17503E-01		0.34250E-01	0.79278E+01	0.19906E-01	0.79079E+01
	4	10.963580	0.10966E+02	0.11004E+02	0.17851E-01		0.55203E-01	0.10949E+02	0.48538E-01	0.10900E+02
	5		0.14204E+02	0.14266E+02	0.16704E-01		0.77899E-01	0.14188E+02	0.78803E-01	0.14109E+02
	6		0.17633E+02	0.17720E+02	0.14963E-01		0.10212E+00	0.17618E+02	0.11053E+00	0.17508E+02
10	0	1.504972	0.14981E+01	0.14981E+01		0.00000E+00		0.14981E+01	-0.33177E-01	0.15312E+01
	1	5.321608	0.53064E+01	0.53064E+01		0.00000E+00		0.53064E+01	-0.75722E-01	0.53821E+01
	2	10.347060	0.10346E+02	0.10351E+02	0.38492E-01		0.43261E-01	0.10307E+02	-0.17112E-01	0.10324E+02
	3	16.090099	0.16099E+02	0.16140E+02	0.49095E-01		0.90490E-01	0.16050E+02	0.50670E-01	0.15999E+02
	4	22.408800	0.22418E+02	0.22510E+02	0.48533E-01		0.14134E+00	0.22369E+02	0.12070E+00	0.22248E+02
	5		0.29216E+02	0.29367E+02	0.44315E-01		0.19523E+00	0.29172E+02	0.19270E+00	0.28979E+02
	6		0.36436E+02	0.36649E+02	0.38936E-01		0.25192E+00	0.36397E+02	0.26688E+00	0.36130E+02
100	0	3.133138	0.31152E+01	0.31152E+01		0.00000E+00		0.31152E+01	-0.77243E-01	0.31924E+01
	1	11.187254	0.11152E+02	0.11152E+02		0.00000E+00		0.11152E+02	-0.17273E+00	0.11325E+02
	2	21.906900	0.21904E+02	0.21914E+02	0.88578E-01		0.98669E-01	0.21815E+02	-0.38546E-01	0.21853E+02
	3	34.182499	0.34203E+02	0.34295E+02	0.11208E+00		0.20390E+00	0.34091E+02	0.11325E+00	0.33978E+02
	4	47.707199	0.47728E+02	0.47934E+02	0.11003E+00		0.31625E+00	0.47618E+02	0.26835E+00	0.47349E+02
	5		0.62293E+02	0.62628E+02	0.99924E-01		0.43474E+00	0.62193E+02	0.42684E+00	0.61766E+02
	6		0.77769E+02	0.78241E+02	0.87418E-01		0.55903E+00	0.77682E+02	0.58950E+00	0.77092E+02
1000	0	6.694221	0.66587E+01	0.66587E+01		0.00000E+00		0.66587E+01	-0.16924E+00	0.68280E+01
	1	23.972210	0.23895E+02	0.23895E+02		0.00000E+00		0.23895E+02	-0.37677E+00	0.24272E+02
	2	47.017342	0.47010E+02	0.47031E+02	0.19357E+00		0.21522E+00	0.46816E+02	-0.83851E-01	0.46900E+02
	3	73.419113	0.73465E+02	0.73664E+02	0.24452E+00		0.44356E+00	0.73220E+02	0.24593E+00	0.72974E+02
	4	102.515999	0.10256E+03	0.10301E+03	0.23969E+00		0.68693E+00	0.10232E+03	0.58210E+00	0.10174E+03
	5		0.13390E+03	0.13463E+03	0.21741E+00		0.94334E+00	0.13368E+03	0.92513E+00	0.13276E+03
	6		0.16721E+03	0.16823E+03	0.19002E+00		0.12121E+01	0.16702E+03	0.12769E+01	0.16574E+03

balance, and (0) for the first kind of balance. For example, $E_n^{(3)}(+)$, $E_n^{(3)}(-)$ and $E_n^{(3)}(0)$ in table 1 represent the maximum and minimum of the third-order perturbation energy for $n \geqslant 2$ states, and the status of the first kind of balance for $n \leqslant 1$ states. As may be seen from table 1, obviously, the first-order approximation $E_n^{(1)}$ is the dominant term of the total energy, namely $E_n^{(1)} \gg E_n^{(p)}$. One can also refer to other theoretical calculations [9–12] for comparison. The balancing frequencies $\Omega_n^{(1)}(-)$, $\Omega_n^{(2)}(+)$, $\Omega_n^{(3)}(-)$ and $\Omega_n^{(3)}(+)$ corresponding to component waves $\Psi_n^{(0)}$, $\Psi_n^{(1)}$ and $\Psi_n^{(2)}$ are listed in table 2 for various coupling parameter values, δ from 0.0001 to 1000. One may see from table 2 that as δ takes the extremely small value of 0.0001, all balancing points $\Omega_n^{(1)}$ to $\Omega_n^{(3)}$ tend to an initial value $c = 1$, likely in equation (43). It is shown in table 3 that the calculated energy values from both standard RS and our ERS are in good agreement and yield accurate estimates of exact energies for small values of the

Table 2. Balance frequency $\Omega_n^{(1)}$ to $\Omega_n^{(3)}$ corresponding to the zeroth- to second-order component waves $\Psi_n^{(0)}$, $\Psi_n^{(1)}$ and $\Psi_n^{(2)}$, respectively, for states $n = 0\text{–}11$ with coupling parameters $\delta = 0.0001\text{–}1000$. The notation (+), (−) and (0) represents the balance state for the corresponding component waves.

n	$\Omega_n^{(1)}(-)$	$\Omega_n^{(2)}(+)$	$\Omega_{n\geqslant2}^{(3)}(-)$	$\Omega_{n\geqslant2}^{(3)}(+)$	$\Omega_n^{(1)}(-)$	$\Omega_n^{(2)}(+)$	$\Omega_{n\geqslant2}^{(3)}(-)$	$\Omega_{n\geqslant2}^{(3)}(+)$
			$\Omega_{n\leqslant1}^{(3)}(0)$				$\Omega_{n\leqslant1}^{(3)}(0)$	
		$\delta = 0.0001$					$\delta = 0.1$	
0	0.1000E+01	0.1000E+01	0.1000E+01		0.1221E+01	0.1231E+01	0.1289E+01	
1	0.1000E+01	0.1000E+01	0.1001E+01		0.1325E+01	0.1335E+01	0.1418E+01	
2	0.1001E+01	0.1001E+01	0.1001E+01	0.1000E+01	0.1443E+01	0.1444E+01	0.1468E+01	0.1576E+01
3	0.1001E+01	0.1001E+01	0.1001E+01	0.1002E+01	0.1545E+01	0.1541E+01	0.1540E+01	0.1713E+01
4	0.1001E+01	0.1001E+01	0.1001E+01	0.1002E+01	0.1635E+01	0.1627E+01	0.1613E+01	0.1826E+01
5	0.1002E+01	0.1002E+01	0.1002E+01	0.1003E+01	0.1715E+01	0.1704E+01	0.1682E+01	0.1924E+01
6	0.1002E+01	0.1002E+01	0.1002E+01	0.1003E+01	0.1787E+01	0.1774E+01	0.1746E+01	0.2011E+01
7	0.1002E+01	0.1002E+01	0.1002E+01	0.1004E+01	0.1854E+01	0.1839E+01	0.1806E+01	0.2091E+01
8	0.1003E+01	0.1003E+01	0.1002E+01	0.1004E+01	0.1916E+01	0.1899E+01	0.1862E+01	0.2165E+01
9	0.1003E+01	0.1003E+01	0.1003E+01	0.1005E+01	0.1974E+01	0.1955E+01	0.1916E+01	0.2233E+01
10	0.1003E+01	0.1003E+01	0.1003E+01	0.1006E+01	0.2028E+01	0.2008E+01	0.1966E+01	0.2298E+01
11	0.1003E+01	0.1003E+01	0.1003E+01	0.1006E+01	0.2080E+01	0.2058E+01	0.2014E+01	0.2358E+01
		$\delta = 1.0$					$\delta = 10$	
0	0.2000E+01	0.2059E+01	0.2207E+01		0.4000E+01	0.4160E+01	0.4480E+01	
1	0.2309E+01	0.2355E+01	0.2566E+01		0.4713E+01	0.4832E+01	0.5295E+01	
2	0.2632E+01	0.2638E+01	0.2725E+01	0.2937E+01	0.5445E+01	0.5461E+01	0.5674E+01	0.6100E+01
3	0.2898E+01	0.2884E+01	0.2897E+01	0.3280E+01	0.6040E+01	0.6007E+01	0.6046E+01	0.6867E+01
4	0.3123E+01	0.3097E+01	0.3076E+01	0.3553E+01	0.6541E+01	0.6481E+01	0.6440E+01	0.7470E+01
5	0.3320E+01	0.3287E+01	0.3243E+01	0.3787E+01	0.6978E+01	0.6899E+01	0.6810E+01	0.7987E+01
6	0.3496E+01	0.3457E+01	0.3398E+01	0.3993E+01	0.7366E+01	0.7275E+01	0.7152E+01	0.8441E+01
7	0.3656E+01	0.3612E+01	0.3542E+01	0.4180E+01	0.7718E+01	0.7616E+01	0.7469E+01	0.8851E+01
8	0.3802E+01	0.3754E+01	0.3677E+01	0.4351E+01	0.8040E+01	0.7931E+01	0.7765E+01	0.9226E+01
9	0.3939E+01	0.3887E+01	0.3802E+01	0.4509E+01	0.8339E+01	0.8222E+01	0.8040E+01	0.9570E+01
10	0.4066E+01	0.4011E+01	0.3920E+01	0.4656E+01	0.8618E+01	0.8495E+01	0.8300E+01	0.9891E+01
11	0.4185E+01	0.4128E+01	0.4032E+01	0.4795E+01	0.8880E+01	0.8750E+01	0.8544E+01	0.1020E+02
		$\delta = 100$					$\delta = 1000$	
0	0.8474E+01	0.8836E+01	0.9525E+01		0.1819E+02	0.1898E+02	0.2046E+02	
1	0.1003E+02	0.1030E+02	0.1130E+02		0.2156E+02	0.2213E+02	0.2430E+02	
2	0.1163E+02	0.1166E+02	0.1213E+02	0.1303E+02	0.2500E+02	0.2507E+02	0.2610E+02	0.2803E+02
3	0.1292E+02	0.1285E+02	0.1294E+02	0.1470E+02	0.2779E+02	0.2763E+02	0.2783E+02	0.3163E+02
4	0.1401E+02	0.1387E+02	0.1379E+02	0.1601E+02	0.3013E+02	0.2985E+02	0.2967E+02	0.3445E+02
5	0.1495E+02	0.1478E+02	0.1459E+02	0.1713E+02	0.3217E+02	0.3180E+02	0.3139E+02	0.3686E+02
6	0.1579E+02	0.1559E+02	0.1533E+02	0.1811E+02	0.3399E+02	0.3356E+02	0.3299E+02	0.3899E+02
7	0.1655E+02	0.1633E+02	0.1602E+02	0.1900E+02	0.3563E+02	0.3515E+02	0.3447E+02	0.4089E+02
8	0.1725E+02	0.1701E+02	0.1666E+02	0.1980E+02	0.3714E+02	0.3662E+02	0.3585E+02	0.4264E+02
9	0.1790E+02	0.1764E+02	0.1725E+02	0.2056E+02	0.3853E+02	0.3798E+02	0.3714E+02	0.4425E+02
10	0.1850E+02	0.1823E+02	0.1781E+02	0.2125E+02	0.3983E+02	0.3925E+02	0.3834E+02	0.4576E+02
11	0.1907E+02	0.1879E+02	0.1834E+02	0.2191E+02	0.4105E+02	0.4045E+02	0.3948E+02	0.4717E+02

coupling parameter $\delta = 0.001$, on the other hand, however, as is to be anticipated, the RS calculation fails completely for the extremely large coupling parameter shown in table 3 as well. The ERS perturbation series of energy levels up to fourth order are shown in table 4 for $n \leqslant 4$ states, where $E_n^{(1)}$ to $E_n^{(4)}$ represent the corresponding perturbation energies, and the notation (+), (−) and (0) indicates the corresponding status of balance. Data in table 4 show that the ERS series converges rapidly because the ratio of $E^{(i+1)}/E^{(i)}$ is quite small. $\Delta E_n^{(3)} = E_n^{(3)}(+) - E_n^{(3)}(-)$ in table 4 indicates energies released such that the second-order component wave passes from unstable to stable. We anticipate that our ERS higher-order approximation should give excellent agreement with the exact answer for the ground state as δ takes extremely large values of $10^4\text{–}10^8$, however, the results are shown in table 6. It does not surprise us that our ERS calculation for the asymptotic limit of δ leads to energy levels which

Table 3. Comparison of the third-order approximate energy levels of ERS calculations with RS results for states $n = 0$–40 with coupling parameters $\delta = 0.001$–10 000. The notation (+), (−) and (0) represents the balance state for the corresponding component waves.

	$\delta = 0.001$		$\delta = 0.1$		$\delta = 1000$		$\delta = 10\,000$	
n	$E_n^{(3)}(-)$	$E_n^{(3),RS}$	$E_n^{(3)}(-)$	$E_n^{(3),RS}$	$E_n^{(3)}(-)$	$E_n^{(3),RS}$	$E_n^{(3)}(-)$	$E_n^{(3),RS}$
0	0.50075E+00	0.50075E+00	0.559E+00	0.570E+00	0.666E+01	0.208E+11	0.137E+02	0.208E+14
1	0.15037E+01	0.15037E+01	0.177E+01	0.191E+01	0.239E+02	0.245E+12	0.508E+02	0.245E+15
2	0.25097E+01	0.25097E+01	0.314E+01	0.396E+01	0.470E+02	0.125E+13	0.101E+03	0.125E+16
3	0.35186E+01	0.35186E+01	0.463E+01	0.758E+01	0.735E+02	0.418E+13	0.158E+03	0.418E+16
4	0.45304E+01	0.45304E+01	0.622E+01	0.142E+02	0.103E+03	0.107E+14	0.221E+03	0.107E+17
5	0.55450E+01	0.55450E+01	0.790E+01	0.258E+02	0.134E+03	0.231E+14	0.288E+03	0.231E+17
6	0.65626E+01	0.65626E+01	0.966E+01	0.450E+02	0.167E+03	0.441E+14	0.360E+03	0.441E+17
7	0.75830E+01	0.75830E+01	0.115E+02	0.749E+02	0.202E+03	0.772E+14	0.436E+03	0.772E+17
8	0.86062E+01	0.86062E+01	0.134E+02	0.119E+03	0.239E+03	0.126E+15	0.515E+03	0.126E+18
9	0.96322E+01	0.96323E+01	0.153E+02	0.182E+03	0.277E+03	0.196E+15	0.597E+03	0.196E+18
10	0.10661E+02	0.10661E+02	0.173E+02	0.268E+03	0.317E+03	0.291E+15	0.682E+03	0.291E+18
11	0.11693E+02	0.11693E+02	0.194E+02	0.383E+03	0.357E+03	0.417E+15	0.770E+03	0.417E+18
12	0.12727E+02	0.12727E+02	0.215E+02	0.533E+03	0.399E+03	0.581E+15	0.860E+03	0.581E+18
13	0.13764E+02	0.13764E+02	0.237E+02	0.724E+03	0.442E+03	0.788E+15	0.953E+03	0.788E+18
14	0.14804E+02	0.14804E+02	0.259E+02	0.963E+03	0.487E+03	0.105E+16	0.105E+04	0.105E+19
15	0.15846E+02	0.15846E+02	0.282E+02	0.126E+04	0.532E+03	0.137E+16	0.115E+04	0.137E+19
16	0.16891E+02	0.16891E+02	0.305E+02	0.162E+04	0.578E+03	0.175E+16	0.125E+04	0.175E+19
17	0.17939E+02	0.17939E+02	0.328E+02	0.205E+04	0.625E+03	0.221E+16	0.135E+04	0.221E+19
18	0.18989E+02	0.18990E+02	0.352E+02	0.256E+04	0.673E+03	0.276E+16	0.145E+04	0.276E+19
19	0.20042E+02	0.20043E+02	0.376E+02	0.317E+04	0.722E+03	0.341E+16	0.156E+04	0.341E+19
20	0.21098E+02	0.21098E+02	0.400E+02	0.388E+04	0.772E+03	0.416E+16	0.166E+04	0.416E+19
21	0.22156E+02	0.22156E+02	0.425E+02	0.470E+04	0.823E+03	0.503E+16	0.177E+04	0.503E+19
22	0.23216E+02	0.23217E+02	0.451E+02	0.565E+04	0.874E+03	0.603E+16	0.188E+04	0.603E+19
23	0.24280E+02	0.24281E+02	0.476E+02	0.673E+04	0.926E+03	0.718E+16	0.200E+04	0.718E+19
24	0.25345E+02	0.25347E+02	0.502E+02	0.797E+04	0.979E+03	0.848E+16	0.211E+04	0.848E+19
25	0.26414E+02	0.26415E+02	0.528E+02	0.936E+04	0.103E+04	0.994E+16	0.222E+04	0.994E+19
26	0.27484E+02	0.27486E+02	0.555E+02	0.109E+05	0.109E+04	0.116E+17	0.234E+04	0.116E+20
27	0.28557E+02	0.28560E+02	0.582E+02	0.127E+05	0.114E+04	0.134E+17	0.246E+04	0.134E+20
28	0.29633E+02	0.29636E+02	0.609E+02	0.147E+05	0.120E+04	0.155E+17	0.258E+04	0.155E+20
29	0.30711E+02	0.30714E+02	0.636E+02	0.169E+05	0.125E+04	0.178E+17	0.270E+04	0.178E+20
30	0.31792E+02	0.31795E+02	0.664E+02	0.193E+05	0.131E+04	0.203E+17	0.282E+04	0.203E+20
31	0.32875E+02	0.32879E+02	0.692E+02	0.220E+05	0.137E+04	0.231E+17	0.295E+04	0.231E+20
32	0.33960E+02	0.33965E+02	0.720E+02	0.249E+05	0.143E+04	0.262E+17	0.307E+04	0.262E+20
33	0.35048E+02	0.35053E+02	0.748E+02	0.282E+05	0.149E+04	0.296E+17	0.320E+04	0.296E+20
34	0.36138E+02	0.36144E+02	0.777E+02	0.317E+05	0.155E+04	0.333E+17	0.333E+04	0.333E+20
35	0.37230E+02	0.37238E+02	0.806E+02	0.356E+05	0.161E+04	0.373E+17	0.346E+04	0.373E+20
36	0.38325E+02	0.38334E+02	0.836E+02	0.398E+05	0.167E+04	0.417E+17	0.359E+04	0.417E+20
37	0.39422E+02	0.39432E+02	0.865E+02	0.444E+05	0.173E+04	0.464E+17	0.372E+04	0.464E+20
38	0.40522E+02	0.40533E+02	0.895E+02	0.494E+05	0.179E+04	0.516E+17	0.385E+04	0.516E+20
39	0.41623E+02	0.41636E+02	0.925E+02	0.548E+05	0.185E+04	0.571E+17	0.399E+04	0.571E+20
40	0.42727E+02	0.42741E+02	0.955E+02	0.606E+05	0.191E+04	0.631E+17	0.412E+04	0.631E+20

are within 99.5% of the exact answer. In the case of the fourth-order perturbation energy $E_n^{(4)}$, the balance status of the third-order component wave $\Psi_n^{(3)}$ becomes complicated. There is a unique maximum of $E_n^{(4)}$ for $n \leqslant 4$ states, and one maximum and one minimum for $n \geqslant 5$ states. If the coupling parameter δ is quite large, however, another maximum is revealed at a distance. Perhaps, it is of great interest because this feature may imply a special physical meaning. Anyway, this is shown in table 5.

The pioneering contribution of ERS perturbation theory was made in this work by extending the balance condition to each component wave. However, some balanced wave may be in an unstable balancing state because of the corresponding perturbation energies taking the maximum. Fortunately, the zeroth-order component waves $\Psi_n^{(0)}$, i.e. zeroth-order waves, definitely are in a stable balance state.

Table 4. ERS perturbation series of energy levels up to the fourth-order approximation for coupling parameters $\delta = 0.001, 0.01, 10^4$ and 10^5.

δ	n	$E_n^{(0)}$	$E_n^{(1)}(-)$	$E_n^{(2)}(+)$	$E_{n \geqslant 2}^{(3)}(-)$	$E_{n \leqslant 1}^{(3)}(0)$	$E_{n \geqslant 2}^{(3)}(+)$	$\Delta E_{n \geqslant 2}^{(3)}$	$E_n^{(4)}(+)$
10^{-3}	0	0.5015E+00	-0.7455E-03	-0.3695E-06	0.0000E+00				0.9241E-13
	1	0.1507E+01	-0.3713E-02	-0.1829E-05	0.0000E+00				0.3214E-10
	2	0.2519E+01	-0.9601E-02	-0.7939E-06	0.3373E-07	0.1255E-05	0.1221E-05		0.7524E-08
	3	0.3537E+01	-0.1836E-01	0.3813E-05	0.7552E-07	0.4814E-06	0.4058E-06		0.5221E-07
	4	0.4560E+01	-0.2994E-01	0.1322E-04	0.1248E-06	0.1424E-05	0.1299E-05		-0.3606E-09
10^{-2}	0	0.5144E+00	-0.7087E-02	-0.3252E-04	0.0000E+00				0.1994E-08
	1	0.1570E+01	-0.3423E-01	-0.1491E-03	0.0000E+00				0.1476E-06
	2	0.2676E+01	-0.8510E-01	-0.5870E-04	0.2049E-04	0.4545E-04	0.2496E-04		0.1870E-04
	3	0.3828E+01	-0.1568E+00	0.2571E-03	0.4102E-04	0.1857E-03	0.1446E-03		0.8667E-04
	4	0.5021E+01	-0.2470E+00	0.8181E-03	0.6042E-04	0.4514E-03	0.3910E-03		0.2350E-03
10^4	0	0.1958E+02	-0.4891E+01	-0.3659E+00	0.0000E+00				0.1913E-02
	1	0.6963E+02	-0.1740E+02	-0.8139E+00	0.0000E+00				0.1760E-01
	2	0.1346E+03	-0.3364E+02	-0.1810E+00	0.4183E+00	0.4649E+00	0.4658E-01		0.3599E+00
	3	0.2095E+03	-0.5235E+02	0.5308E+00	0.5282E+00	0.9576E+00	0.4294E+00		0.7312E+00
	4	0.2921E+03	-0.7300E+02	0.1256E+01	0.5176E+00	0.1483E+01	0.9649E+00		0.1153E+01
10^5	0	0.4217E+02	-0.1054E+02	-0.7890E+00	0.0000E+00				0.4127E-02
	1	0.1500E+03	-0.3750E+02	-0.1754E+01	0.0000E+00				0.3794E-01
	2	0.2900E+03	-0.7248E+02	-0.3902E+00	0.9019E+00	0.1002E+01	0.1003E+00		0.7758E+00
	3	0.4512E+03	-0.1128E+03	0.1144E+01	0.1139E+01	0.2064E+01	0.9254E+00		0.1576E+01
	4	0.6292E+03	-0.1573E+03	0.2707E+01	0.1116E+01	0.3195E+01	0.2080E+01		0.2485E+01

Table 5. The minima and maxima of the fourth-order perturbation energy for higher excited states of $n \geqslant 5$, and the coupling parameter $\delta = 0.001$ and $10\,000$.

	$\delta = 0.001$				$\delta = 10\,000$					
n	$E_n^{(4)}(+)$	$\Omega_n^{(4)}(+)$	$E_n^{(4)}(-)$	$\Omega_n^{(4)}(-)$	$E_n^{(4)}(+)$	$\Omega_n^{(4)}(+)$	$E_n^{(4)}(-)$	$\Omega_n^{(4)}(-)$	$E_n^{(4)}(+)$	$\Omega_n^{(4)}(+)$
5	-0.831E-09	1.0429	-0.176E-08	1.0174	0.164E+01	90.30	-0.953E-01	70.4738	-0.707E-01	68.00
6	-0.104E-08	1.0172	-0.434E-08	1.0205	0.220E+01	95.44	-0.133E+00	74.4295	-0.641E-01	70.33
7	-0.519E-09	1.0195	-0.709E-08	1.0232	0.286E+01	99.50	-0.139E+00	77.7848	-0.183E-01	73.77
8	0.167E-08	1.0218	-0.736E-08	1.0256	0.364E+01	104.20	-0.100E+00	80.6946	0.396E-01	76.58
9	0.730E-08	1.0250	-0.552E-09	1.0276	0.457E+01	107.99	-0.563E-02	83.1959	0.117E-00	79.67

Table 6. The asymptotic values of up to fourth-order approximation for the ground state of ERS calculations for extremely large coupling parameter $\delta = 10^4$ to 10^8, and comparison with the exact ratio of 0.667 99 by numerical calculation.

δ	$E_0^{(4)}(+)$	$E_0^{(4)}(+)/\delta^{1/3}$	$\Omega_0^{(4)}(+)$	$E_0^{(3)}(0)$	$E_0^{(3)}(0)/\delta^{1/3}$	$\Omega_0^{(3)}(0)$
10000.00000	0.1432E+02	0.66482	0.4529E+02	0.1432E+02	0.66473	0.4406E+02
100000.0000	0.3085E+02	0.66457	0.9757E+02	0.3084E+02	0.66448	0.9490E+02
1000000.000	0.6645E+02	0.66452	0.2102E+03	0.6644E+02	0.66443	0.2045E+03
10000000.00	0.1432E+03	0.66451	0.4529E+03	0.1431E+03	0.66442	0.4405E+03
100000000.0	0.3084E+03	0.66451	0.9756E+03	0.3084E+03	0.66442	0.9490E+03

It is no surprise that a number of these results promise the special satisfaction of ERS perturbation with applications to problems which are connected to strong interactions. Our results in this work indicate that our ERS approximation may be valid even in the strong-coupling limit.

The numerical calculations performed in this work yielded relative errors $\leqslant 10^{-5}$, and have been carried out on a personal computer (386/DX) with software written by the author.

Acknowledgments

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